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Anomalous Behavior of the Zero Field Susceptibility of the Ising Model on the Cayley Tree

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It is found that the zero field susceptibility χ of the Ising model on the Cayley tree exhibits unusually weak divergence at the critical point T_C. The susceptibility amplitude is found to diverge at T_C proportionally to the tree generation level n, while the behavior of χ is otherwise analytic in the vicinity of T_C, with the critical exponent $\gamma = 0$.

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It has been well established[1−4] that the Ising model on the Cayley tree exhibits no spontaneous order (zero magnetization in zero field), while the susceptibility diverges for temperatures lower than the critical value $T_C = 2k_B^{-1}J/\ln[(\sqrt{B}+1)/(\sqrt{B}-1)]$, where k_B is the Boltzmann constant, J is the nearest-neighbor interaction parameter, and B is the tree branching number (coordination number minus one). Above T_C, it was found through several different *approximation schemes*[1−4] that the susceptibility asymptotically behaves as

$$\chi = \frac{\beta\,(u+1)^2}{(1-B\,u^2)}, \tag{1}$$

where $\beta = 1/k_BT$ and $u = \tanh(\beta J)$. Expanding the above expression around $t \equiv (T - T_C)/T_C$ one obtains

$$\chi = \frac{\sqrt{B}+1}{2J\sqrt{B}\left(\sqrt{B}-1\right)}\, t^{-1} + O\left(t^0\right) \tag{2}$$

corresponding to the "classical" value of the critical exponent $\gamma = 1$. The same value $\gamma = 1$ was also found[5,6] for the Bethe lattice (interior of the Cayley tree), with the (higher) critical temperature $T_{BC} = 2k_B^{-1}J/\ln[(B+1)/(B-1)]$.

On the other hand, in a recent work[7] the present authors have established the *exact closed form expression* for the zero-field susceptibility $\chi(n,T)$ of the Ising model on a Cayley tree of arbitrary generation n. Our result[7] confirmed previous findings[1,3,4] concerning susceptibility high temperature behavior ($T > T_C$), but it also sheds new light on its singular behavior in the low temperature region $T \le T_C$. In this work we perform a finite size scaling analysis of the zero field susceptibility, paying special attention to its singular behavior in the thermodynamic limit, at and below T_C. In particular, it turns out that the singular behavior of χ at T_C, as displayed by the exact formula, is quite different from that deduced from the approximate asymptotical expression (1) used in previous works[1,3,4]. More precisely, we find that the terms of the order $\sim t^{-1}$ exactly cancel out, and it is shown that the susceptibility displays what may be termed a "divergent coincident singularity", in Fisher's classification[8], with critical exponent $\gamma = 0$.

For simplicity, we consider here only the Cayley tree with the branching number $B = 2$, the generalization to arbitrary B being straightforward. We further consider a single n-generation branch of a Cayley tree, composed of two $(n-1)$-generation branches connected to a single initial site, with the Hamiltonian

$$\mathcal{H} = -J\sum_{\langle nn\rangle} S_iS_j - H\sum_i S_i, \tag{3}$$

where H is the external magnetic field, $S_i = \pm 1$ is the spin at site i, $\langle nn\rangle$ denotes summation over the nearest-neighbor pairs, and J denotes the coupling constant as before. The n-generation branch consists of $N_n = 2^{n+1} - 1$ spins, the 0-generation branch being a single spin. The recursion relations for the partition function of any two consecutive generation branches were found by Eggarter[9] to be

$$Z_{n+1}^{\pm} = y^{\pm 1}\left[x^{\pm 1}Z_n^{+} + x^{\mp 1}Z_n^{-}\right]^2, \tag{4}$$

where $x \equiv \exp(\beta J)$, $y \equiv \exp(\beta H)$, and Z_n^+ and Z_n^- denote the branch partition functions restricted by fixing the initial spin (connecting the two $(n-1)$-generation branches) into the $\{+\}$ and $\{-\}$ position, respectively.

As the nonlinear coupled recursion relations (4) can be iterated to yield a closed form expression[9] only in the zero field case, several rather sophisticated approximation schemes[1–4] have been devised to elaborate the field dependence of the partition function, and deduce (among other quantities of interest) the limiting behavior of the zero field susceptibility *in the high temperature region*, as given by equation (1).

On the other hand, only recently[7] we have arrived at the exact analytical expression for the zero field susceptibility, by considering the recursion relations for the *field derivatives* of the partition function, which can be iterated in the limit $H \longrightarrow 0$ to yield corresponding closed form expressions. Zero field susceptibility of a Cayley tree of *arbitrary* generation n, as a function of temperature, was thus found[7] to be

$$\chi_n = \frac{\beta}{2^{n+1}-1}\left[\frac{(u+1)^2\ 2^{n+1}}{1-2\,u^2} + \frac{u^2\ 2^{n+1}\ \left(2u^2\right)^{n+1}}{(2\,u^2-1)(2\,u-1)^2} + \frac{2^{n+2}\ u^{n+2}}{(2\,u-1)^2} + \frac{2\,u^2-1}{(2\,u-1)^2}\right]. \tag{5}$$

In the limit $n \longrightarrow \infty$ the first term ($h_n^{(1)}$) inside the square brackets on the right hand side of (5) recovers formula (1), and represents the dominant term for temperatures $T > T_C$. The second term ($h_n^{(2)}$) is dominant in the low temperature region $T < T_C$, while the last two terms ($h_n^{(3)}$ and $h_n^{(4)}$) become negligible in the thermodynamic limit for all temperatures, and are therefore relevant only for finite size systems.

Up to this point everything matches the previous results and conclusions of other authors[1–4], therefore, it came as somewhat of a surprise to find that upon expansion in power series around $t = 0^{\pm}$ the terms proportional to t^{-1} *exactly cancel out, independent of the system generation level (therefore, also in the thermodynamic limit), on both sides of the critical point.* It will be shown in the remainder of this paper that this fact leads to quite different conclusions about critical behavior of χ, from those that follow from the approximate expression (1).

Expanding the four individual sequential terms, $h_n^{(1)}$, $h_n^{(2)}$, $h_n^{(3)}$ and $h_n^{(4)}$, inside the brackets on the right hand side of (5), in the vicinity of T_C, and retaining only the terms of the order t^{-1}, it is found

$$\begin{aligned} h_n^{(1)} &= \frac{\left(\sqrt{2}-1\right)^2\ 2^n}{\sqrt{2}\,K_C}\ t^{-1} + O(t^0),\\ h_n^{(2)} &= -\frac{\left(\sqrt{2}-1\right)^2\ 2^n}{\sqrt{2}\,K_C}\ t^{-1} + O(t^0),\\ h_n^{(3)} &= O(t^0),\\ h_n^{(4)} &= O(t^1) \end{aligned} \tag{6}$$

where $K_C \equiv J/k_BT_C = \ln\left(1+\sqrt{2}\right) \sim 0.881374$. It is seen that the terms proportional to t^{-1} exactly cancel out for arbitrary n, for both positive and negative t. Retaining additional powers in t, for large n one obtains

$$\chi_n = n\ K_C \frac{\left(1+\sqrt{2}\right)^2}{2J}\left[1 - t\ n\ K_C\ \frac{1}{\sqrt{2}} + O(t^2)\right], \tag{7}$$

which is valid on both sides of the critical point for $|t| << 1/n$ (since t is an independent small parameter, this condition can be fulfilled for arbitrarily large n). It is seen from (7) that in the thermodynamic limit χ_n diverges at T_C proportionally to tree generation level n, rather then as t^{-1} as follows from approximate formula (1), *and is otherwise analytic* in t. It therefore follows that the susceptibility critical exponent is equal to $\gamma = 0$, while the *amplitude* diverges as $n \longrightarrow \infty$, demonstrating behavior that may be termed[8] "divergent coincident singularity". It should be noted that a Cayley tree of generation $n = 77$ has a number of spins corresponding to the Avogadro's number, while for $n \sim 270$ the number of spins corresponds to the estimated number of hadrons in the observable Universe. Therefore, the above susceptibility exhibits an extremely weak singularity.

To illustrate the behavior of susceptibility in the vicinity of T_C in Fig. 1 we show the function χ_n/n, as given by formula (5), for several system sizes $n = 64, 128, 256, 512$ and 1024. All the shown curves χ_n/n intersect at $k_BT_C/J = 1/\ln\left(1+\sqrt{2}\right) \sim 1.134593$, exhibiting a finite value $\ln\left(1+\sqrt{2}\right)\ \left(1+\sqrt{2}\right)^2/2 \sim 2.568511$. Note that these correction terms turn already imperceptible on the scale of the graph.

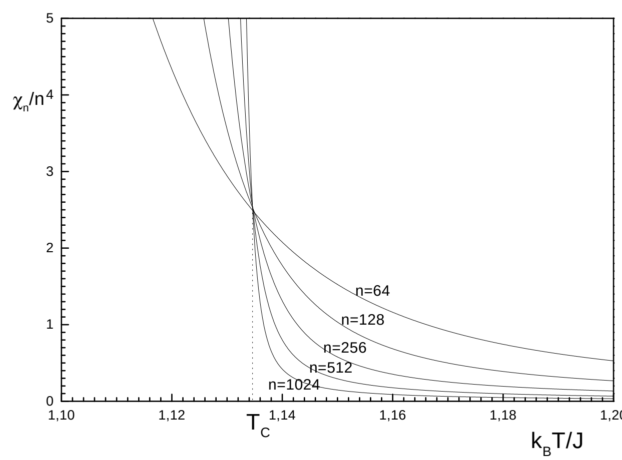

FIG. 1. Zero field susceptibility as a function of temperature, calculated using formula (5), for several system sizes $n = 32, 64, 126, 256$ and 1024. Crossing of the curves χ_n/n at T_C demonstrates the unusually weak singular behavior.

In the temperature region below T_C the second term ($h_n^{(2)}$) inside the brackets on the right hand side of (5) becomes dominant, and for large n susceptibility is given by

$$\chi_n = \beta \frac{u^2\ \left(2u^2\right)^{n+1}}{(2\,u^2-1)(2\,u-1)^2}. \tag{8}$$

It is seen that the susceptibility diverges as $\left(2u^2\right)^{n+1}$,

the divergence becoming stronger as temperature is lowered. Scaling of susceptibility for different system sizes is obtained by taking the logarithm and then dividing by the generation level n. To demonstrate this scaling behavior, in Fig. 2 we show the function $\ln \chi_n / (n+1)$ for several system sizes, together with the corresponding limiting function $\ln(2u^2)$. It is seen that the displayed finite size curves become hardly distinguishable from the limiting function for $0 < T < T_C$.

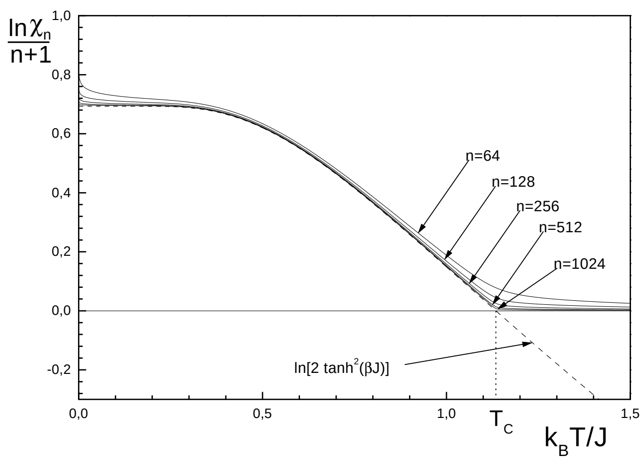

FIG. 2. Finite size scaling of the zero field susceptibility in the temperature region $0 < T < T_C$. The shown curves are calculated using formula (5), for several system sizes $n = 32, 64, 126, 256$ and 1024. The dotted line represents the limiting curve $\ln\left[2\tanh^2(\beta J)\right]$.

Finally, as T approaches zero, the multiplicative term β takes over and susceptibility diverges trivially as T^{-1}, for all generation levels.

In conclusion, the exotic structure of the Cayley tree, with its infinite dimension and finite order of ramification, gives rise to rather unusual thermodynamic behavior. The overall scaling behavior of susceptibility is governed by scaling of its amplitude, rather then distance from the critical point. In particular, contrary to the conclusions drawn from the previous works[1,3,4] using the approximate formula (1), at T_C the susceptibility displays an extremely weak singularity, with amplitude diverging proportionally to the tree generation level n (as n increases to infinity in the thermodynamic limit), with the critical exponent $\gamma = 0$.

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